Boundedness of varieties of log general type : Section 3

Main theorem Fix $n \in \mathbb{N}$ and a set $I \subseteq [0,1]$ which satisfies DCC.

$\mathcal{D}$ set of projective log canonical pairs (X,B) such that $\dim X = n$, $\mathrm{coeff}(B) \subseteq I$.

Then there exist $\delta > 0$, an integer $m > 0$ st

1) $\{ \mathrm{vol}(X, K_X + B) \mid (X,B) \}$ satisfies DCC

2) If $\mathrm{vol}(X, K_X + B) > 0$ then $\mathrm{vol}(X, K_X+B) \geq d$

3) If $K_X + B$ is big then $\phi_{m(K_X+B)}$ is birational.

Theorem 1 (Boundedness on anticanonical volume)

Let $\mathcal{D}$ be the set of KH pairs (X,B) st X is projective $\dim X = n$ $K_X + B \equiv 0$, $\mathrm{coeff}(B) \subseteq I$.

Then $\exists M(n, I) > 0$ st

$\mathrm{vol}(X, -K_X) < M$ for any $(X, B) \in \mathcal{D}$.

<u>Theorem 2</u> (Effective birationality)

Let $\mathcal{B}$ be the set of log canonical pairs st $\dim X = n$,

$\mathrm{coeff}(B) \subseteq I$, $K_X + B$ is big.

Then there is $m(n, I) > 0$ st

$\phi_m (K_X + B)$ is birational for $m \geq m(n, I)$

for $(X, B) \in \mathcal{B}$.

<u>Theorem 3</u> (ACC for numerically trivial pairs) There is a <u>finite</u> subset

$I_0 \subseteq I$ st for all (X, B)

$$\left.\begin{array}{l} (1)\ (X, B) \text{ lc pair} \\ (2)\ \mathrm{coeff}(B) \subseteq I \\ (3)\ K_X + B \equiv 0 \end{array}\right\} \Rightarrow \mathrm{coeff}(B) \subseteq I_0.$$

Theorem 4 (ACC for LCT)

There is $\delta > 0$ st if

(1) (X, B) is n-dim log pair coeff$(B) \subseteq I$

(2) (X, ϕ) is klt for some $\phi \geq 0$.

(3) (X, B') is lc and $B' \geq (1-\delta) B$

Then (X, B) is lc.

Proof of main theorem

Theorem 2 $\Rightarrow$ (3).

For (1): Fix $M > 0$. Consider

(X, B) st $\mathrm{vol}(X, K_X + B) \leq M$.

(3) + DCC for I + condition on volume

$\Rightarrow$ family is log birationally bounded.

By previous results $\Rightarrow$ vol satisfies DCC.

(1) $\Rightarrow$ (2) obvious.

Boundedness of anticanonical volume

Proof (X, B) klt $\quad K_X + B \equiv 0$.

We may assume X is $\mathbb{Q}$-factorial

Suppose $\mathrm{vol}(-K_X) > 0$.

If $x \in X$ there is $G \sim_{\mathbb{R}} -K_X$

$$\mathrm{ord}_x G > \frac{1}{2} \mathrm{vol}(-K_X)^{1/n}$$

$$\mathrm{lct}(X, G) < \frac{2n}{\mathrm{vol}(-K_X)^{1/n}}$$

So $(X, (1-\delta)B + \delta G)$ is lc not klt

for some $\delta < \dfrac{2n}{\mathrm{vol}(-K_X)^{1/n}}$

$\Phi \overset{\text{def}}{=} (1-\delta) B + \delta G$.

Extract the unique lc place

$$
\begin{array}{cccc}
\nu : & X' & \longrightarrow & X \\
& \cup & & \cup \\
& E & \longrightarrow & Z
\end{array}
$$

$$
\begin{cases}
K_{X'} + B' + aE = \nu^*(K_X + B) \\
K_{X'} + \phi' + E = \nu^*(K_X + \phi)
\end{cases}
$$

Run $K_{X'} + \phi' \equiv_X -E'$ MMP

$$
\begin{tikzcd}
X' \arrow[r, dashed, "\phi"] & X'' \arrow[d, "\pi"] \\
& W
\end{tikzcd}
$$

Mori fiber space

Pushforward to X''.

$$\begin{cases} K_{X''} + \phi'' + E'' \text{ is plt} \\ K_{X''} + \phi'' + E'' \equiv 0 \\ E'' \text{ ample over } W. \end{cases}$$

$$\begin{cases} (K_{X''} + B'' + E'')|_{E''} = K_{E''} + B_{E''} \\ \\ (K_{X''} + \phi'' + E'')|_{E''} = K_{E''} + \phi_{E''} \end{cases}$$

(1) coeff $(B_{E''})$ are DCC

(2) $K_{E''} + \phi_{E''}$ is klt

(3) $\phi_{E''} \geq (1-\delta) B_{E''}$ since
$\phi'' \geq (1-\delta) B''$

If $\mathrm{vol}(-K_X) \gg 1$ then $\delta \ll 1$

By Thm 4_{n-1} (ACC for LCT)

$\Rightarrow K_{E''} + B_{E''}$ is lc.

$\phi'' > (1-\delta) B''$ and also

B'' is not contained in fibers of π

$\Rightarrow K_{X''} + (1-\eta) B'' + E'' \equiv_W 0$

for some $0 < \eta < \delta$

$\mathrm{Diff}_{E''}((1-\eta)B'') \geq (1-\eta) B_{E''}$

Suppose $\eta \to 0$.

Coeff $(\mathrm{Diff}_{E''}(1-\eta)B'')$ are dcc

by Thm 3_{n-1} they are finite

So $\eta \not\to 0$ So $\delta \not\to 0$

So $\mathrm{vol}(-K_X) \not\to \infty$. $\square$

Birational boundedness

Theorem Assume Thm 2_{n-1} + Thm 1_n

Then there is a constant $\beta < 1$

st if (X, B) is n-dim lc pair

$K_X + B$ big coeff $(B) \subseteq I$ then

$$\lambda = \inf \{ t \in \mathbb{R} \mid K_X + tB \text{ is big} \} \leq \beta$$

Pf By contradiction take $\lambda_i \to 1$.

Step 1 Produce sequence of

$\mathbb{Q}$-factorial klt pairs (Y_i, Γ_i)

st coeff $(\Gamma_i) \subseteq I$, $-K_{Y_i}$ ample

$K_{Y_i} + \Gamma_i \equiv 0$. $\dim Y_i \leq n$.

We may assume (X, B) log smooth.

Take $D \sim_{\mathbb{Q}} K_X + B$

Consider $K_X + \mu B + \varepsilon D \sim (1+\varepsilon)(K_X + \lambda B)$

$\mu < \lambda$ as $\varepsilon \to 0$ $\mu \to \lambda$

Since $\mu B + \varepsilon D$ is big, we can run $K_X + \mu B + \varepsilon D$ - MMP.

$f : X \dashrightarrow X'$ $K_{X'} + \lambda B'$ is nef k lt

Now we can run $K_{X'} + \mu B'$ - MMP

End up : $g : X' \dashrightarrow X''$

$$\begin{array}{c} X'' \\ \downarrow \pi \\ Z \end{array}$$

π is $K_{X''} + \mu B'' + \varepsilon D''$ - trivial contraction

st $\varepsilon D''$ is ample $/ Z$.

So $-K_{X''}$ is ample $/Z$.

$K_{X''} + \lambda B''$ is klt

$K_{X''} + \lambda B'' \equiv_Z 0$

Take $(Y, \Gamma) = (F, B''|_F)$

<u>Step 2</u> Take $\nu_i : Y_i' \to Y_i$

be a log resolution.

$D_i = (\Gamma_i)_{red}$ $\Gamma_i' = (\nu_*^{-1}) \Gamma_i + Ex(\nu)$

D_i' ---

<u>Easy</u>: $K_{Y_i'} + \Gamma_i'$ and $K_{Y_i'} + D_i$

are big

By Thm 1_n $\mathrm{vol}(Y_i, \lambda_i \Gamma_i) < C$

$\Longrightarrow$ (Y_i', D_i') are log birationally bounded.

some work

+ thm 2

By DCC for volumes in the log birationally bounded case, we have $\mathrm{vol}(Y_i', K_{Y_i'} + \Gamma_i') \geq \delta$.

$$\delta \leq \mathrm{vol}(Y_i', K_{Y_i'} + \Gamma_i')$$
$$\leq \mathrm{vol}(Y_i, K_{Y_i} + \Gamma_i)$$
$$\leq \underset{(\lambda)}{\text{expression}} \cdot \mathrm{vol}(Y_i, \lambda_i \Gamma_i)$$

$$\delta \leq \text{expression}(\lambda) \cdot C$$

Thm 2_{n-1} + Thm 1_n $\Rightarrow$ Thm 2_n

Pf We just showed $\exists\, \gamma < 1$ st $K_X + \gamma\Delta$ is big. Fix q st $(1-\gamma) \cdot \min(I) > \frac{1}{q}$

Consider $I_0 = \left\{ \frac{1}{q}, \cdots, \frac{q-1}{q}, 1 \right\}$

$\exists \Delta_0$ st $\gamma \Delta \leq \Delta_0 \leq \Delta$

$\mathrm{coeff}(\Delta_0) \subseteq I_0$.

$K_X + \Delta_0$ is big

$\Rightarrow$ conclude by effective birationality in the case of finitely many coefficients.

ACC for numerically trivial pairs

Thm 3_{n-1} + Thm 2_n $\Rightarrow$ Thm 3_n

Proof

Let $J = D(I)$. J DCC.

Apply Thm 3_{n-1} to J.

We get J_0.

Consider $I_1 \subseteq I$ the finite set defined by

$$I_1 = \left\{ i \in I \;\middle|\; \frac{m-1+f+ki}{m} \in J_0 \quad f \in D(I) \right\}$$

Let (X, B) n-dim log pair st

$K_X + B \equiv 0$. $\operatorname{coeff}(B) \subseteq I$.

We may assume (X, B) dlt.

$B = \sum b_i B_i$.

If B_i intersects a component

S of $\lfloor B \rfloor$, let

$K_S + \Theta = (K_X + B)|_S$

$\operatorname{coeff}(\Theta) \subseteq J$.

(S, Θ) is lc $K_S + \Theta \equiv 0$.

By thm 3_{n-1} $\operatorname{coeff}(\Theta) \subseteq J_0$.

Let P be an irreducible component

of $B_i|_S$

Then $\operatorname{mult}_p \# = \dfrac{n-1+f+kb_i}{n}$

So $b_i \in I_1$

$\boxed{\text{Conclusion: if } B_i \text{ meets } S \text{ then } b_i \in I_1}$

If $b_i \notin I_1$ then $B_i \cap \lfloor B \rfloor = \emptyset$

If there are any such, choose one.

Run $K_X + B - b_i B_i$ - MMP

with scaling of an ample divisor.

$K_X + B - b_i B_i \equiv - b_i B_i$ ← not contracted

we get

$$\begin{tikzcd} X \arrow[r, dashed] & X' \arrow[d] \\ & Z \end{tikzcd}$$

Since B_i does not intersect $\lfloor B \rfloor$ none of the components of $\lfloor B \rfloor$ get contracted.

If $\dim Z > 0$ then restrict to general fiber and conclude by Thm 3_{n-1}.

So assume $\dim Z = 0$.

$\rho(X') = 1 \implies \lfloor B \rfloor = 0$.

$\implies (X, B')$ is klt.

We have reduced to the case

1) (X, B) klt
2) $\rho(X) = 1$.

Let $m(n, I)$ given by Thm 2a

We want to show that for

any $1 \le l \le m$

$$\left| I \cap \left[\frac{l-1}{m}, \frac{l}{m} \right) \right| \le 1$$

$$\Rightarrow |I| \le m$$

Suppose by contradiction $i_1 < i_2$

Take (X, B) klt $\rho(X) = 1$

$$B = \sum b_i B_i \qquad b_1 = i_1$$

Take a log resolution

$\nu : X' \to X$, consider

$$(X', B') \quad B' = \nu_*^{-1}\left(B + (i_2 - i_1) B_1\right) + Ex(\nu).$$

$$K_{X'} + B' = \nu^*(K_X + B) + (i_2 - i_1)\nu_*^{-1} B_1 + F$$

is big

$\Longrightarrow$ $\phi_m(K_{X'} + B')$ is birational

$$\Longrightarrow \quad m\left(K_X + \frac{\lfloor m \nu_* B' \rfloor}{m}\right) \text{ big}$$

By the choice of i_1, i_2

$$B \geq \frac{\lfloor m \nu_* B' \rfloor}{m}$$

<u>Contradiction</u>: $K_X + B \equiv 0$

$$K_X + \frac{\lfloor m \nu_* B' \rfloor}{m} \text{ big}$$

$\square$

Acc for lct

Go to the lct.

Extract lc place.

Do adjunction.

By adding a bit of an ample one gets a big log canonical divisor, then use the fact that pseudoeffective thresholds do not accumulate to 1.